M.R P. 1015
S.C. 11/220
30th November, 1956

The air flow over a ridge - the effects of the upper boundary and high - level conditions

by

G. A. Corby and J. S. Sawyer

(Meteorological Office)

AIR MINISTRY

METEOROLOGICAL RESEARCH COMMITTEE

At the time of issue this report has not been published. Any author desiring to quote this paper should refer to it as "A paper of the Meteorological Research Committee (London). M.R.P. No. 1015 by Messrs. G.A. Corby and J.S. Sawyer
A copy is available in the Library of the Meteorological Office."

M.R.P. 1015
S.C.II/220
30th November, 1956

AIR MINISTRY
METEOROLOGICAL RESEARCH COMMITTEE

The air flow over a ridge - the effects of the upper boundary and high-level conditions

by

G. A. Corby and J.S. Sawyer (Meteorological Office)

(In this paper the symbol $\rightleftharpoons$ is used to denote "equals the real part of".)

Introduction

A differential equation can be derived to represent the flow of air over a ridge. If attention is confined to the two-dimensional problem and small perturbations only are considered, the equation can be linearised and takes on a simple form.

The problem of obtaining the flow over a particular ground profile consists of determining the solution of the equation subject to suitable boundary conditions. The boundary condition at the ground is imposed directly by the requirement that the velocity there should be everywhere parallel to the ground, but additional conditions are also required to be applied either at some artificial boundaries introduced to limit the problem in the vertical and horizontal or at great heights and large horizontal distances. Only if adequate conditions of this nature are imposed does the problem become mathematically determinate.

It is usual to provide adequate boundary conditions at large horizontal distances by requiring that the disturbance to the flow should tend to zero for large distances upstream and should remain finite for large distances downstream. These requirements are based on an appeal to physical realism.

In the present paper we consider what boundary conditions should be applied at great heights and whether the introduction of an arbitrary upper boundary gives a justifiable approximation to the flow. The results of some computations using model airstreams including the stratosphere are presented to illustrate the effect on the flow of different characteristics in the airstream at high levels.

The physical problem

In the treatment of the problem by Scorer (1949), the flow is defined by ψ which is the departure of the stream function from its value in the undisturbed stream, and the differential equation obtained for small perturbations in two-dimensional flow takes the form:

$$\frac{\partial^2\psi}{\partial z^2} - \left(\frac{g}{c^2} + \beta\right)\frac{\partial\psi}{\partial z} + \left(l^2 - k^2\right)\psi = 0 \qquad (1)$$

where z = height measured upwards

g = gravity

where c = velocity of sound

β = static stability in the form $\theta^{-1}\partial\theta/\partial z$, θ being the potential temperature

$\ell^2 = \frac{g\beta}{U^2} - \frac{1}{U}\frac{\partial^2 U}{\partial z^2}$, a function of the stability and wind profiles in the undisturbed stream

U = undisturbed horizontal wind speed

k = 2π / (wavelength of the disturbance).

We wish to consider the behaviour of this equation for large values of z under conditions typical of the stratosphere. For simplicity we shall assume that in the stratosphere U and T are constant.

Then $$\left(\frac{g}{c^2} + \beta\right) = -\frac{1}{\rho}\frac{\partial \rho}{\partial z} = \frac{g}{RT} \qquad 2$$

Also $$\ell^2 = \frac{g\beta}{U^2} = \frac{\gamma-1}{\gamma}\frac{g}{RTU^2} \qquad 3$$

Hence in a uniform stratosphere Eq. 1 becomes

$$\frac{\partial^2\psi}{\partial z^2} - \frac{g}{RT}\frac{\partial\psi}{\partial z} + \left(\frac{\gamma-1}{\gamma}\frac{g^2}{RTU^2} - k^2\right)\psi = 0 \qquad 4$$

The solutions, corresponding to a known harmonic corrugation in the lower boundary of wavelength $2\pi/k$, have the form $\psi = e^{mz}.e^{ikx}$

where $$m = \frac{1}{2}\frac{g}{RT}\left\{1 \pm \sqrt{1-4M}\right\} \qquad 5$$

and $$M = \frac{R^2T^2}{g^2}\left(\frac{\gamma-1}{\gamma}\frac{g^2}{RTU^2} - k^2\right) \qquad 6$$

As m may be real or complex these solutions represent waves whose amplitude increases or decreases exponentially (or is constant) with height or which vary harmonically in the vertical with increasing or decreasing (or constant) amplitude. In any given case the types of solutions which appear will depend on k, T and U.

Wholly real values of m

We first consider the case of real roots for m. This occurs when $4M \leq 1$, or, in terms of the wavelength, when $k^2 \geq g^2/R^2T^2\left(\frac{\gamma-1}{\gamma}\frac{RT}{U^2} - \frac{1}{4}\right)$

The solution $\psi = e^{mz}e^{ikx}$ represents a wave which increases in amplitude with increasing height if $m > 0$ and the wave will decrease with height if $m < 0$. The wave fronts will necessarily be vertical.

There are thus three possibilities:

(i) If $M > 0$, both solutions represent waves which increase in amplitude with height.

(ii) If $M < 0$, one solution increases with height and the other decreases.

(iii) If $M = 0$, one solution has constant amplitude with height and the other increases.

In terms of k , the condition $M \geqslant 0$ becomes

$$k^2 \leqslant \frac{\gamma-1}{\gamma} \frac{g^2}{RTU^2} \qquad 7$$

Before deciding, however, which solutions are relevant to our problem, we must note that as the density of the air decreases with height, the energy of the wave disturbance may decrease with height even when the amplitude increases with height. It would seem more appropriate to exercise discrimination in terms of the energy of the disturbance than in terms of the amplitude.

The kinetic energy of the wave motion is proportional to $\rho(u^2+w^2)$, where u and w are the perturbation velocities in the x and z directions

$$\text{i.e.,} \quad K.E. \propto \rho\left(\left(\frac{\partial\psi}{\partial z}\right)^2 + \left(\frac{\partial\psi}{\partial x}\right)^2\right)$$

$$\propto \rho e^{2mz}$$

$$\propto e^{(2m-g/RT)z}$$

Thus the kinetic energy of the disturbance decreases with height provided $m < \frac{1}{2}g/RT$. Hence of those solutions under consideration, viz., real values of m , each pair comprises one solution having energy increasing with height and one solution with energy decreasing with height. The amplitude of the latter solution will itself decrease with height only if $M < 0$. There is, therefore, a narrow band of wavelengths corresponding to $0 < M < \frac{1}{4}$, for which one of the solutions, although having increasing amplitude with height, nevertheless has decreasing energy; but this band proves to involve only a very small range of wavelengths. For solutions within this band of wavelengths the amplitude will become so great at some heights that the basic assumption of perturbation theory (namely small amplitude) will become invalid. Nevertheless there is no reason to expect that, if the complete non-linear equations could be solved for these high levels, the order of magnitude of the energy would be altered. Consequently we accept as appropriate to our problem all solutions of Eq. 1 having energy decreasing with height. This is achieved by selecting the negative sign in the expression for m given in Eq. 5.

Complex values of m

If $4M > 1$, that is for wavelengths longer than a critical limit, m will be complex and the solutions take the form

$$\psi = e^{gz/2RT} \cdot e^{i(kx \pm Nz)}$$

$$\propto \rho^{-\frac{1}{2}} \cdot e^{i(kx \pm Nz)} \qquad 8$$

where $N = \frac{1}{2}g/RT\left\{4M-1\right\}^{\frac{1}{2}}$

$$= \left(l^2 - k^2 - \tfrac{1}{4}g^2/R^2T^2\right)^{\frac{1}{2}}$$

Hence

$$u = \frac{\partial \psi}{\partial z} \propto -\tfrac{1}{2} P^{-3/2} \frac{\partial P}{\partial z} \propto P^{-\frac{1}{2}}$$

$$w = \frac{\partial \psi}{\partial x} \propto P^{-\frac{1}{2}}$$

The kinetic energy of the disturbance is proportional to $\rho(u^2 + w^2)$ i.e., to ρP^{-1} = constant. The kinetic energy does not, therefore, decrease with height for any wavelength in the range under consideration and the total kinetic energy above any level would appear to be infinite. However, such solutions occur only for the flow over an infinite series of sinusoidal troughs and ridges, and it can be demonstrated that for the flow over a ridge extending only a finite distance up - or down - stream the energy above any level is finite and tends to zero as that level is raised indefinitely. This difference arises because, as the energy is proportional to the square of the velocity, the energy at the various wavelengths in the range is not additive and the interference between waves of different wavelengths prevents the appearance of infinite energy.

The above discussion indicates that it is not possible to devise an adequate upper boundary condition for this range of wavelengths on the basis of energy considerations. The problem remains as to which of the two solutions given by Eq. 8, or what mixture of the two solutions, should be adopted.

We now consider the flow when a rigid upper boundary exists. This eliminates any ambiguity and renders the flow fully determinate. We shall then examine the flow as the rigid lid is removed towards infinity. If the flow at finite heights tends to a limit under these circumstances, the inference would be very strong that this limit is the appropriate choice for the unbounded atmosphere in nature.

Suppose our two-dimensional ridge is specified by the Fourier Integral:

$$\zeta = \int_0^\infty e^{ikx} \cdot F(k)\, dk \qquad 9$$

where $F(k)$ is real and necessarily bounded for all positive k, and let us assume that a rigid lid exists at $z = h$.

The general solution of our differential equation, as applied to a uniform airstream, is

$$\psi = e^{\frac{1}{2}\frac{g}{RT}z} \left(A e^{iNz} + B e^{-iNz} \right) e^{ikx} \qquad 10$$

where $N = \left(l^2 - k^2 - \frac{1}{4} g^2/R^2T^2 \right)^{\frac{1}{2}}$

and A, B are constants which may be complex.

Put $l^2 - \frac{1}{4} g^2/R^2T^2 = L^2$. We are only concerned with the case of $L^2 > 0$, since otherwise N would not be complex for any real value of k.

By introducing the upper and lower boundary conditions into Eq. 10 we derive as the expression for the displacement of a streamline at height z in the undisturbed stream:

$$\zeta_z = \int_0^\infty e^{\frac{1}{2}\frac{g}{RT}z} \left(\cos Nz - \sin Nz \cdot \cot Nh \right) e^{ikx} \cdot F(k)\, dk$$

The part of the integral over the range $k > L$ corresponds to wholly real values of m, for which the upper boundary condition has been shown to be determinate from energy considerations. We need only consider, therefore, the range $k < L$.

We note that if $Lh > \pi$ the integrand has one or more poles in this range of k and as h becomes large, the number of poles increases. Thus the effect of the lid is to introduce lee waves, which become infinite in number as $h \to \infty$. It can be shown that individually they become infinitesimal in amplitude for large h except for the special case of $k = 0$ and $Lh = n\pi$, when the amplitude is indeterminate. However the shape of the mountain ridge is not sensibly altered by defining it as $\int_{\epsilon}^{\infty} e^{ikx} F(k)\,dk$, where ϵ is small and positive, and this avoids the singularity.

Following the method established by Lamb (1932) for similar lee wave problems, we integrate by deforming the contour in the complex plane of k, noting that the integrand has no poles off the axes (see Fig. 1). That part of the displacement arising from the range $0 < k < L$ is thus given by:

$$p'_x = \left\{ e^{\frac{1}{2}\frac{g}{RT}z} \int_{C_1} (\cos Nz - \sin Nz \cdot \cot Nh)\, e^{ikx} \cdot F(k)\,dk \right.$$

$$\left. + \pi i \left(\sum \text{residues at the poles} \right) \right\} \quad \text{for } x > 0 \qquad 11$$

$$\text{and} = \left\{ e^{\frac{1}{2}\frac{g}{RT}z} \int_{C_2} (\cos Nz - \sin Nz \cdot \cot Nh)\, e^{ikx} \cdot F(k)\,dk \right.$$

$$\left. - \pi i \left(\sum \text{residues at the poles} \right) \right\} \quad \text{for } x < 0 \qquad 12$$

We now appeal to physical realism and annul the waves upstream by adding $\pi i \left(\sum \text{residues} \right)$ on both sides of the mountain, so obtaining as the practical solution in our problem:

$$p'_x = \left\{ e^{\frac{1}{2}\frac{g}{RT}z} \int_{C_1} (\cos Nz - \sin Nz \cdot \cot Nh)\, e^{ikx} \cdot F(k)\,dk \right.$$

$$\left. + 2\pi i \left(\sum \text{residues at the poles} \right) \right\} \quad \text{for } x > 0 \qquad 13$$

$$\text{and} = \left\{ e^{\frac{1}{2}\frac{g}{RT}z} \int_{C_2} (\cos Nz - \sin Nz \cdot \cot Nh)\, e^{ikx} \cdot F(k)\,dk \right.$$

$$\text{for } x < 0 \qquad 14$$

We must now consider whether this integral tends to a finite limit as $h \to \infty$. Let $k = k_1 + i k_2$ along contour C_1

i.e., $\sqrt{L^2 - (k_1 + i k_2)^2} = N = N_1 + i N_2$, say,

or $N_1^2 - N_2^2 = L^2 - k_1^2 + k_2^2$

and $N_1 N_2 = -k_1 k_2$.

Now $N = +L$ when $k_1 = k_2 = 0$ and N_1 and N_2 must be of opposite sign. Thus in the first quadrant N_1 is +ve and N_2 is -ve. Hence $\cot Nh$ is of the form $\cot(N_1 - iN_2)h$ along C_1.

It can be shown that

$$\lim_{h \to \infty} \left\{ \cot (N_1 - iN_2)h \right\} = i$$

≠ Hence

$$\lim_{h\to\infty}\int_{k_1}(\cos Nz - \sin Nz \cdot \cot Nh)\, e^{ikx}\cdot F(k)\,dk$$
$$= \int_0^L e^{-iNz+ikx}\, F(k)\,dk$$

We now require to evaluate $\lim_{h\to\infty}(2\pi i \sum \text{residues})$.

The residue at a pole where $k = k'$, $N = N'$ is:

$$\frac{N'}{hk'}(\sin N'z)\, e^{ik'x}\cdot F(k')$$

The poles occur where $Nh = 0, \pi, 2\pi, \ldots\ldots n\pi$

i.e., where $k = \left(L^2 - n^2\pi^2/h^2\right)^{\frac{1}{2}}$

$r = 0, 1, 2, \ldots\ldots n$

Put $Lh = \pi(n+\alpha)$ where $0 \le \alpha < 1$

Then the poles occur where $k = L\left[1 - r^2/(n+\alpha)^2\right]^{\frac{1}{2}}$

Hence $\lim_{h\to\infty}\sum$residues =

$$\lim_{n\to\infty}\left\{e^{\frac{1}{2}\frac{g}{RT}z}\sum_{r=0}^{n} e^{ik'x}\cdot F(k')(\sin N'z)\left(\frac{+L}{\pi(n+\alpha)^2\left\{1-r^2/(n+\alpha)^2\right\}^{\frac{1}{2}}}\right)\right\}$$
$$= \frac{1}{\pi}e^{\frac{1}{2}\frac{g}{RT}z}\int_0^L e^{ikx}\cdot F(k)\sin Nz\; dk$$

Thus $2\pi i\,(\sum \text{residues}) = 2i\, e^{\frac{1}{2}\frac{g}{RT}z}\int_0^L e^{ikx}\cdot F(k)\sin Nz\,dk$

Substituting in Eqs. 13 and 14 we see that

$$\varphi' = e^{\frac{1}{2}\frac{g}{RT}z}\int_0^L e^{+iNz+ikx}\cdot F(k)\,dk \qquad 15$$

It would appear, therefore, that the proper solution is obtained by taking $B = 0$ in the general solution of Eq. 10. The same conclusion was reached by different methods by other authors, e.g., Queney (1947), Palm (1952), although Scorer (1949) adopted the choice $A = 0$. Scorer discusses the difficulties attendant upon this choice in a later paper (Scorer, 1954).

≠ It is easily verified that it is here justified to take $\lim_{h\to\infty}$ inside the integral sign.

The present treatment can be interpreted physically as follows. If there is a rigid upper boundary, then the flow of a uniform airstream over a ridge comprises a symmetrical disturbance over the ridge plus a large number of lee waves having wavelengths in the range $2\pi/L$ to ∞ . The waves tend to interfere away from the ridge and as the lid is removed to infinity they annul completely for large x . They contribute, however, a finite disturbance in the neighbourhood of the ridge for positive x and the whole flow tends to a limit which is asymmetrical. It seems inescapable that this must be the solution appropriate to the unbounded atmosphere.

It is of interest to consider briefly the nature of the solutions obtained for a two-layer model in which the value of L is constant in the lower layer and zero in the upper layer. Under these conditions m is not complex in the upper layer for any value of k ; accordingly there is no indeterminacy at high levels. As the depth of the lower layer is increased, more and more lee waves appear for values of k in the range $0 < k < L$ and these become infinite in number as the interface is removed to infinity. Once again the flow at finite heights tends to the asymmetrical limit corresponding to the choice $B = 0$ in Eq. 10. The proof is somewhat more lengthy but the artificiality of a rigid upper boundary is avoided.

The characteristics of certain model airstreams

In studying the character of the flow with specific model airstreams we shall use the simpler form of Eq. 1 as employed by Scorer (1949), who shows that that the character of the solution is not affected by omission of the central term in Eq. 1. The equation then becomes

$$\frac{\partial^2 \psi}{\partial z^2} + (l^2 - k^2)\psi = 0 \qquad 16$$

If we wished we could allow approximately for the neglected middle term by multiplying the displacements of the streamlines deduced from Eq. 16 by the factor $(\rho_0/\rho)^{\frac{1}{2}}$, where ρ is the density and suffix 0 refers to the ground.

Suppose our model airstream comprises layers of depth a_1 , a_2 , ------ in each of which l is constant, the values of l being l_1 , l_2 , ------ and the suffixes being numbered from the ground upwards.

Then, if we adopt the upper boundary condition established above, the flow over the mountain profile given by

$$\psi_0 = \int_0^\infty e^{ikx} F(k)\, dk$$

takes the form:

$$\psi_z = \int_0^\infty (\cos \nu_1 z + f(k) \sin \nu_1 z)\, e^{ikx} F(k)\, dk \qquad 17$$

where $\nu_1 = +(l_1^2 - k^2)^{\frac{1}{2}}$

The flow specified by Eq. 17 applies to the lower layer and for convenience we have assumed that the undisturbed wind speed U is constant with height, i.e., that the variations of l with height arise solely due to variations of static stability. The function $f(k)$ takes the following forms for two, three and four-layer models:

2 layers

$$f(k) = \frac{f_1}{f_2} = \frac{\nu_1 t_1 + i\nu_2}{\nu_1 - i\nu_2 t_1} \qquad 18$$

3 layers

$$f(k) = \frac{f_1}{f_2} = \frac{\nu_2(\nu_1 t_1 + \nu_2 t_2) + i\nu_3(\nu_2 - \nu_1 t_1 t_2)}{\nu_2(\nu_1 - \nu_2 t_1 t_2) - i\nu_3(\nu_1 t_2 + \nu_2 t_1)} \qquad 19$$

4 layers

$$f(k) = \frac{f_1}{f_2} = \frac{\nu_2(\nu_3 - i\nu_4 t_3)(\nu_1 t_1 + \nu_2 t_2) + \nu_3(i\nu_4 + \nu_3 t_3)(\nu_2 - \nu_1 t_1 t_2)}{\nu_2(\nu_3 - i\nu_4 t_3)(\nu_1 - \nu_2 t_1 t_2) - \nu_3(i\nu_4 + \nu_3 t_3)(\nu_1 t_2 + \nu_2 t_1)} \qquad 20$$

where $\nu_n = +(l_n^2 - k^2)^{\frac{1}{2}}$

and $t_n = \tan(\nu_n a_n)$

The values of k giving rise to lee waves are the roots of the equations $f_2 = 0$ and, in the lower layer, that part of the displacement of a streamline due to the lee waves appropriate to a root $k = k^*$ is given by:

$$\zeta_z = 2\pi i \left[\frac{f_1}{\partial/\partial k (f_2)} \sin \nu_1 z \; e^{ikx} \cdot F(k) \right]_{k = k^*} \qquad 21$$

Similar expressions can be derived for a multilayer model in which a lid exists at any specified level.

Eqs. 18-21 can be used to compute the properties of the lee waves which arise in any two, three or four-layer model airstream, the awkward quantity $\partial/\partial k(f_2)$ being obtained numerically in the course of the computations. The behaviour in the vertical of the lee waves in the layers above the lowest is obtained by applying considerations of continuity at the interfaces.

(i) Two-layer airstream

We shall first consider a simple two-layer airstream of the type which can contain lee waves and then examine how these lee wave characteristics are modified as different conditions are specified at high levels. We take a lower stable layer, 1 km deep, in which $l^2 = 4.0\ \text{km}^{-2}$ and suppose that in the upper layer which extends to infinity $l^2 = 0.3\ \text{km}^{-2}$. The appropriate lee wave equation has only one real root viz: $k = 0.76\ \text{km}^{-1}$, corresponding to a lee wavelength of 8.3 km.

In order to assign an absolute magnitude to the amplitude we specify the mountain profile as that given by

$$\zeta_0 = h b \int_0^\infty e^{ikx - kb}\, dk$$

and we take the height of the ridge, h, as 0.1 km. The amplitude of the

lee wave will vary with b, the so-called "half-width" parameter, and to particularise we take $b = k^{-1}$ being the value which excites the largest amplitude lee waves. In a discussion of the amplitude of lee waves, Corby and Wallington (1956) regarded the mountain having $b = k^{-1}$ as the optimum width for the particular lee wavelength, and likened the effect near this value to resonance.

Under these conditions the variation with height of the lee wave amplitude as determined from Eq. 21 is that illustrated in Fig. 2(a). The variation of amplitude with height follows the usual pattern for such an airstream in that a maximum amplitude is attained in the lower layer, where the amplitude variation in the vertical is sinusoidal; in the upper layer the amplitude falls off exponentially. For convenience, the value of the lee wave amplitude at the level where the maximum is achieved is illustrated for other values of b in Fig. 2(b).

It is of interest to consider here the effect of introducing a rigid lid at, say, 10 km above the interface or roughly at tropopause level. It is found that the lee wave at $k = 0.76\ \text{km}^{-1}$ still appears, without sensible change in its properties, but that there is an additional lee wave at $k = 0.415\ \text{km}^{-1}$. The variation with height of the two waves is indicated in Fig. 3(a) where the amplitude of each is depicted for its own optimum-width mountain. Fig. 3(b) enables the amplitudes for any particular mountain to be compared. The second wave, due to the presence of the lid, is noteworthy in that it has its maximum amplitude in the upper layer. This and several other computations suggest that, although the imposition of a lid at great height is theoretically a useful artifice, it would not give a good approximation to the flow of an unbounded atmosphere unless imposed at very great height indeed. In the present case, with the lid at height 11 km, only one additional lee wave is induced and clearly we should require a sufficiently large number to annul almost completely by interference away from the mountain. The point would be relevant if one considered using a rigid lid as an upper boundary condition when treating the problem by numerical integration of the differential equation for the flow, or when considering model experiments.

(ii) Four-layer airstream

We now consider a four-layer airstream in which the two lower layers of depth 1 and 10 km have the same values of l^2 used in the two-layer example. They are surmounted by a third layer of depth 30 km in which $l^2 = 6.0\ \text{km}^{-2}$; this is intended to represent a stable stratosphere. Above this we assume the lapse rate to be adiabatic ($l^2 = 0$). Such a model is not completely unrealistic, for the upper half of the ozonosphere is believed to be a region of active convection (see Scorer, 1953). Furthermore, the presence of the upper layer with $l^2 = 0$ means that the disturbance necessarily decreases with height in that layer over the whole range of wavelengths.

As the terms involving $\tan \nu_3 a_3$ in Eq. 20 oscillate rapidly for changing values of k, the wavelength equation $f_2 = 0$ has a large number of roots. Indeed, it is found that there are 26 nodal surfaces in the solution for $k = 0$, indicating the existence of 26 lee wavelengths. These can be conveniently put into three groups, viz., those in the ranges $0 < k < l_2$, $l_2 < k < l_1$ and $l_1 < k < l_3$. No lee waves occur for $k > l_3$. All the lee waves have sinusoidal variation of amplitude through the stable stratosphere and exponential decrease of amplitude in the uppermost layer.

There are two lee waves in the first group ($0 < k < l_2$). These are of appreciable amplitude at low levels and have even larger amplitude in the upper troposphere and stratosphere especially if we bear in mind the additional factor $(\rho_0/\rho)^{\frac{1}{2}}$ which for simplicity has been omitted at this stage. Formally the solutions of the equation $f_2 = 0$ corresponding to these lee waves arise because of the decrease in stability between the second and upper layers and they would not appear, for example, if we made $l_{1+} = l_2$. However, some similarity will be noted between these waves and the solution which was introduced into the two-layer model by the rigid lid at 11 km, and it can be shown that the form of these solutions in the lower layers is largely determined by

the large increase of stability in passing from the second to the third layer, corresponding to the increase in stability at the tropopause. Although, if no less stable layer is introduced at great heights no infinite trains of lee waves appear as solutions, nevertheless $|f(k)|$ in Eq. 17 becomes large for values of k near that corresponding to lee waves with a lid and it can be shown that oscillations of the same type appear in the solution down-stream of the obstacle but that these die away slowly down-stream.

The first root in the second group, viz., that having the smallest k, occurs very nearly at $k = 0.76$ km^{-1} and the amplitude variation through the troposphere is very closely similar to that of the wave obtained from the two-layer model; indeed for practical purposes the waves can be regarded as identical. The next wave of the group has an amplitude in the lower layer only about 1/1000 of that at $k = 0.76$ km^{-1} and subsequent waves have even smaller amplitudes. The waves in the third group are also of negligible amplitude. Many of these additional waves do, however, attain their maximum amplitude in the stratosphere although even there the absolute values are negligible.

The amplitude and its variation with height are illustrated for the three main waves in Fig. 4(a). As before, the absolute magnitude of the amplitude is depicted in each case for the appropriate optimum-width mountain, and Fig. 4(b) may be used to compare the amplitudes in the lower layer for any other specific mountain width.

If the upper layer is omitted and a lid is imposed at the top of the stratosphere, once again the main lee wave near $k = 0.76$ km^{-1} appears without significant change in its properties. There are also two lee waves of longer wavelength (smaller k) having similar properties to those with the four-layer model, whilst all the other lee waves have negligible amplitude at all heights.

For most purposes the lee waves having the greatest interest are those which attain their maximum amplitude in the lower troposphere and which die away upwards. The unbounded two-layer airstream first considered had one lee wavelength of this type and the subsequent computations suggest that, provided the same tropospheric conditions are maintained, this lee wave appears without perceptible change in its properties whatever conditions are assumed at high levels. Thus lee waves of this type do not appear to be at all sensitive to changes in the wind or stability profiles at high levels. That this is not a fortuitous result arising because of the particular l^2 profile adopted, but is true in general, can be deduced by a study of Eq. 20. It also appears that lee waves may arise as a result of the change of stability in the region of the tropopause. These are similar to the solutions obtained if a rigid lid is imposed at a level corresponding to the tropopause but for proper representation of these the stability at greater heights must be considered. The introduction of a layer of neutral stability or a lid at a very high level is a useful artifice to make these lee waves appear as infinite wave trains rather than wave trains of finite extent, thereby making the mathematical treatment easier.

The solution obtained for $k = 0$

The device of using the number of nodal surfaces in the solution for $k = 0$ as a means of determining the number of lee wavelengths was referred to earlier. We will mention another possible application of this solution.

In a problem such as that of orographic rainfall we are not concerned with the smaller scale gravity waves with wavelengths of the order of 10 km or less, but with those comparable in length with the horizontal scale of a barrier such as the Welsh mountains. On the other hand such a barrier is still small enough to allow the neglect of oscillations about the geostrophic flow without changing the nature of the problem. Accordingly, the main features of the flow will be adequately represented by Eq. 16. Here l^2 varies from 0 to 10 km^{-2} or more; zero values occur when the atmosphere has neutral static stability, and the more commonly occurring lapse rates above the surface layers usually lead to values of $l^2 > 0.25$ km^{-2}.

If we consider the mountain ridge whose profile is represented by Eq. 22, then clearly the amplitude of the Fourier components becomes small (e.g., $< 1/12$ of the maximum) for $k > 2.5/b$. Thus for a mountain half-width of 20 km the contribution of components with $k > 0.12$ is small and we are primarily concerned with k^2 in the range 0 to 0.015. It follows that a general idea of the large scale flow can be obtained by putting $k = 0$ in Eq. 16 and solving it in the form

$$\frac{\partial^2 \psi}{\partial z^2} + l^2 \psi = 0 \qquad 23$$

where l is now a function of z.

Since the solution near the ground must be independent of the value of l at great heights we consider a case in which l is constant above a specified upper level.

We require a solution of the form

$$\psi = F(z) \cdot e^{ikx} \qquad 24$$

where $F(z)$ is a complex function of z $\{F(z) = F_1(z) + iF_2(z)\}$ and in order that this should have the correct form in the upper layer with constant l, we require that at any level in that layer, $\partial F/\partial z = ilF$. This provides the upper boundary condition.

The solution (24) may be assumed to hold approximately throughout the range of k in which we are interested. Then the flow at level z over a mountain with profile:

$$\zeta_0 = h b \int_0^\infty e^{ikx - kb}\, dk$$

is given by

$$\zeta_z = h b \int_0^\infty (F_1 + iF_2)\, e^{ikx - kb}\, dk$$

or

$$\psi_z = \frac{hb}{b^2 + x^2} (l F_1 - x F_2) \qquad 25$$

The approximation implied by putting $k = 0$ in this way, was used by Scorer (1949) when obtaining the part of the flow above the hill (as distinct from the lee waves). The point being made here is that it is a satisfactory and realistic approximation when studying the large-scale character of the flow over a barrier such as the Welsh mountains or the Pennines in connection with orographic rainfall. In the case of l constant with height, both the symmetric and asymmetric components in the displacement (Eq. 25) have a vertical wavelength of $2\pi/l$. We, therefore, require a small value of l if the upslope motion on the windward slope is to continue as an upslope motion to considerable heights. If the air mass is very stable (statically) l will be large and we would expect to find descending air at no very great height above the windward slope. These aspects bear intimately on the problem of what conditions favour heavy orographic rain.

Although only approximate, Eq. 25 gives a specific indication of the vertical variation and asymmetry of the flow over a barrier such as the Pennines or the Welsh mountains. In layers with static stability a measurable disturbance of the temperature field would be expected and some observable effects should be apparent in the distribution of cloud layers.

Horizontal flights by aircraft well above such a barrier might therefore produce direct evidence of the validity of the treatment and in particular of the appropriateness of the upper boundary condition which has been deduced and adopted here.

Conclusions

In the theory of the airflow over mountains, further light is thrown on the problem of what choice of upper boundary condition should be adopted by studying the limit to which the flow tends as a rigid lid, or some other upper condition which renders the problem fully determinate, is removed to infinity. This device strongly suggests that, provided we are right in assuming that a stationary solution does exist, then in the uppermost layer and over the range of wavelengths for which solutions with inclined wave fronts occur, the proper procedure is to reject the solutions giving wave fronts inclined in the downstream direction and to retain those inclined in the upstream direction. This conclusion is consistent with that reached by including small friction to achieve determinacy or by obtaining the asymptotic limit of a time-varying solution.

When calculating the character of the flow in actual cases, the familiar type of lee wave which has a maximum amplitude in the lower troposphere and which dies away above, is not at all sensitive to the conditions assumed at high levels. It would appear, however, that lee waves which have their largest amplitude in the upper troposphere and stratosphere can occur in certain airstreams depending primarily on the change of stability near the tropopause; these are of somewhat longer wavelength, and for their proper study require consideration of stability and wind in the stratosphere.

REFERENCES

Lamb, Sir H.	1932	Hydrodynamics, 6th Edn. arts 242-246.
Queney, P.	1947	Theory of perturbations in stratified currents with applications to airflow over mountain barriers. Chicago Univ. D. Misc. Rep. No. 23.
Scorer, R. S.	1949	Theory of waves in the lee of mountains. Quart. J.R. Met. Soc., 75, p.41.
Palm, E.	1952	On the formation of surface waves in a fluid flowing over a corrugated bed and on the development of mountain waves. Astrophys. Norvegica, 5, No. 3.
Scorer, R. S.	1953	Air motion in the upper atmosphere. Nature, 172, p.138.
Corby, G. A. and Wallington, C. E.	1956	Airflow over mountains - the lee wave amplitude. Quart. J.R. Met. Soc., 82, p.266.
Scorer, R.S.	1954	Theory of airflow over mountains III - Airstream characteristics. Quart. J.R. Met. Soc., 80, p.417.

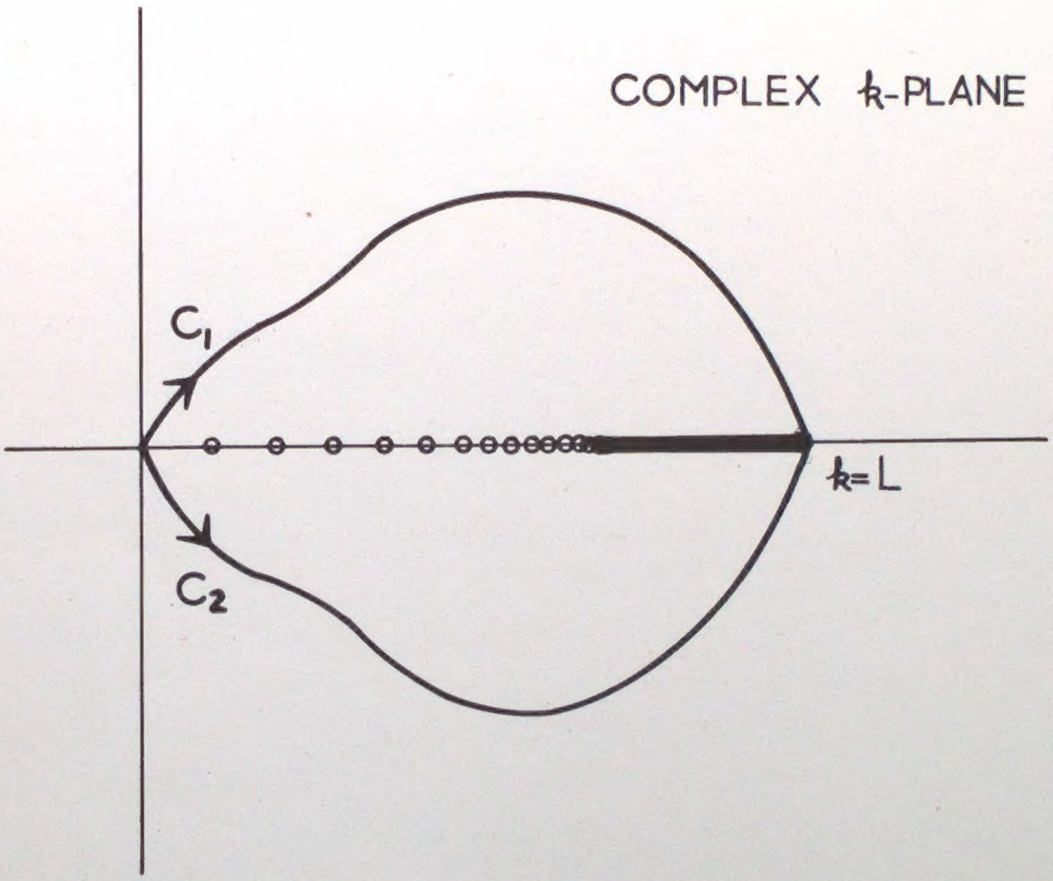


Fig. 1. Contours used for contour integration.

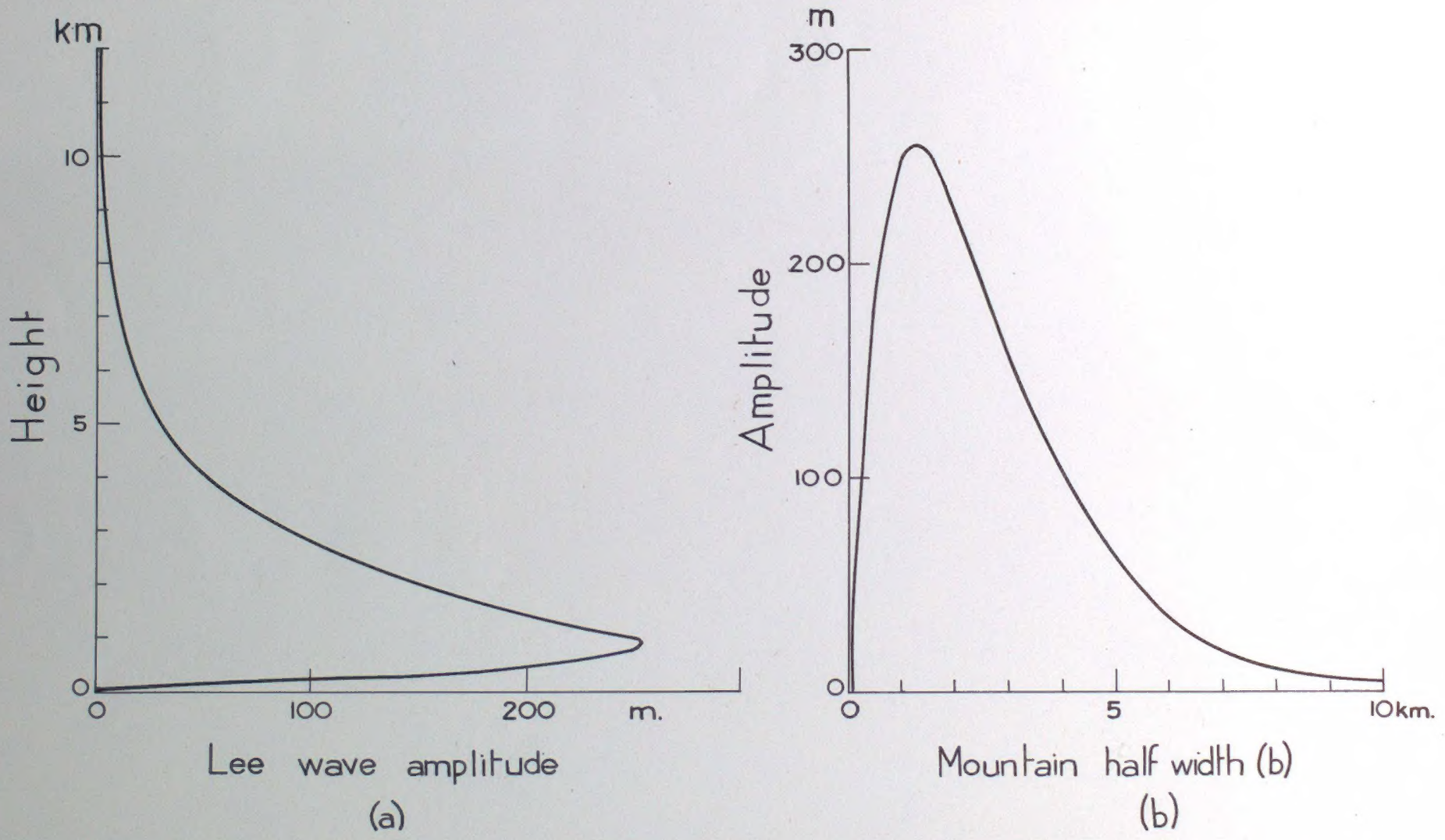


Fig. 2. The properties of the lee-wave ($k = 0{\cdot}76\ km^{-1}$) for a two-layer airstream extending to infinity and flowing over a ridge of height 100 m. In (a) is depicted the variation of amplitude with height for the optimum-width ridge, and in (b) the variation with mountain width of the maximum amplitude in the lower layer.

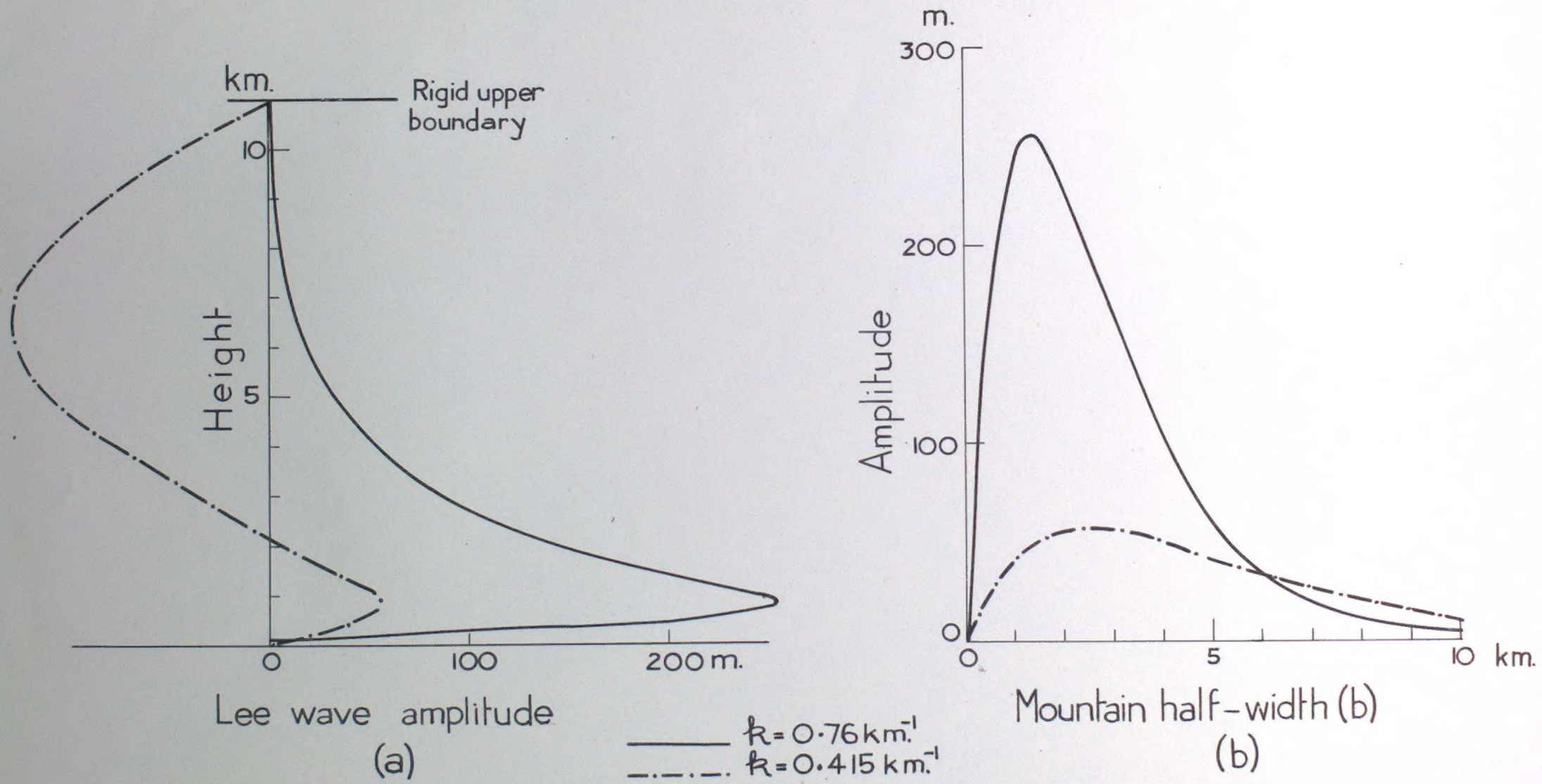
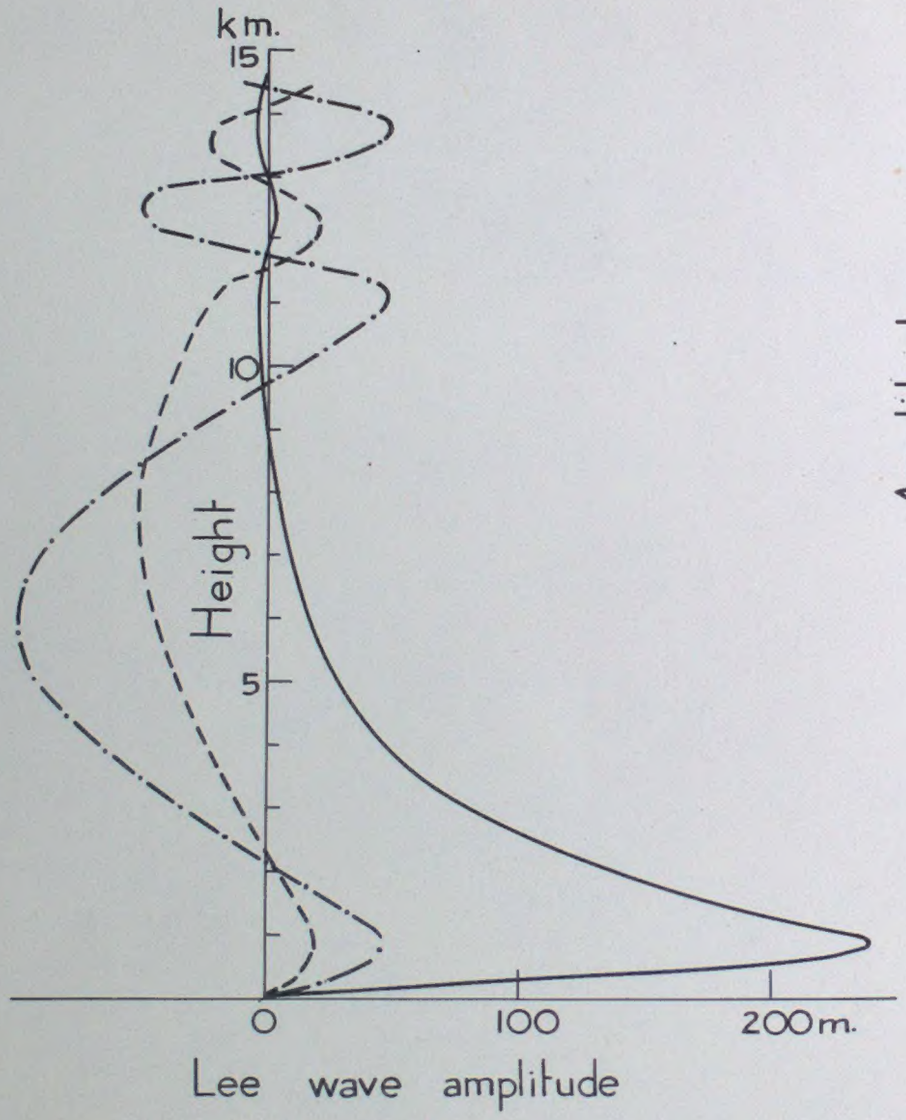


Fig. 3. The properties of the two lee-waves for a two-layer airstream bounded by a rigid lid and flowing over a ridge of height 100 m. In (a) is depicted the variation of amplitude with height, each lee-wave being associated with its optimum-width ridge, and in (b) the variation with mountain width of the maximum amplitude in the lower layer.



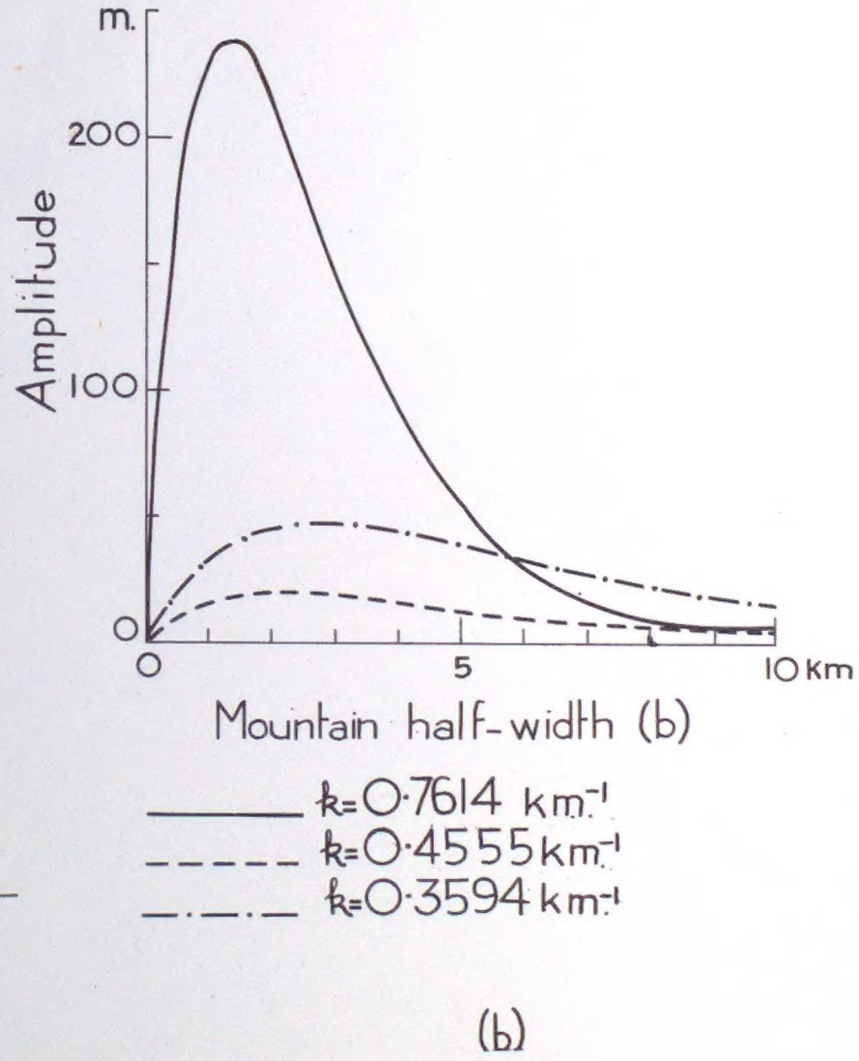


Fig. 4. The properties of the three principal lee-waves for a four-layer airstream extending to infinity and flowing over a ridge of height 100 m. In (a) is depicted the variation of amplitude with height, each lee-wave being associated with its optimum-width ridge, and in (b) the variation with mountain width of the amplitude in the lower layer.